Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of APPLIED MATHEMATICS AND MECHANICS
ELSEVIER
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 891–899

WEAK EVASION OF A GROUP OF COORDINATED EVADERS†

A. I. BLAGODATSKIKH
Izhevsk
email: aiblag@mail.ru

(*Received* 17 *May* 2004)

The problem of the group pursuit of a group of evaders who use the same control, in which the manoeuvrability of the evaders is higher, is considered. A position control is constructed which ensures a weak evasion (that is, the non-coincidence of the geometrical coordinates, speeds, accelerations and so forth) of all the evaders.

Evasion problems for a single evader who possesses a greater manoeuvrability, with discrimination of the pursuers, were considered earlier in [1, 2]. Problems of the group pursuit of a single evader by different types of pursuers subject to a discrimination condition for the evader were presented in [3, 4]. This paper extends the results, obtained earlier in [5] for simple motions of the evaders, to the case of more general motions.

1. FORMULATION OF THE PROBLEM

A differential game Γ of $n + m$ players is considered in the space R^ν ($\nu \geq 2$): n of these players are pursuers $P_1, P_2, \dots, P_n$ and m are evaders $E_1, E_2, \dots, E_m$ with the laws of motion and the initial conditions (when $t = 0$)

$$\begin{gathered} x_i^{(p)} = u_i, \quad \|u_i\| \leq 1; \quad y_j^{(q)} = \upsilon, \quad \|\upsilon\| \leq \gamma, \quad \gamma \in (0, 1), \quad p > q \geq 1 \\ x_i^{(\alpha)}(0) = X_i^\alpha, \quad \alpha \in P, \quad y_j^\beta(0) = Y_j^\beta; \quad X_i^\beta \neq Y_j^\beta, \quad \beta \in Q \end{gathered} \tag{1.1}$$

Henceforth,

$$i \in I = \{1, 2, \dots, n\}, \quad j = 1, 2, \dots, m, \quad c = 1, 2$$

$$P = \{0, 1, \dots, p-1\}, \quad Q = \{0, 1, \dots, q-1\}$$

Definition 1. The controls $u_i(t)$ and $\upsilon(t)$ from the class of measurable functions which satisfy the restrictions from (1.1) are called permissible controls.

Definition 2. In the game Γ, there is a weak evasion if a permissible control

$$\upsilon(t) = \upsilon(t, x_i^{(\alpha)}(t), \alpha \in P, y_j^{(\beta)}(t), \beta \in Q)$$

is found for any permissible controls $u_i(t)$ such that

$$x_i^{(\beta)}(t) \neq y_j^{(\beta)}(t), \quad \beta \in Q \quad \text{for all} \quad t \in [0, \infty)$$

†*Prikl. Mat. Mekh.* Vol. 69, No. 6, pp. 993–1002, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.11.011

The actions of the evaders can be treated as follows: there is a centre which, using the quantities $\{x_i^{(\alpha)}(t), \alpha \in P, y_j^{(\beta)}(t), \beta \in Q\}$, at each instant of time $t \geq 0$, chooses the same control $\upsilon(t)$ for all the evaders E_j.

2. THE CASE WHEN $m = 1$

We will construct a permissible control $\upsilon(t)$ which ensures a weak evasion in the problem with a single evader E_1. In this action, we will omit the subscript $j = 1$ in relations (1.1).

It follows from the possibility of a weak evasion when $\nu = 2$, that is, the case of a plane, that a weak evasion is also possible when $\nu > 2$. In fact, if $\nu > 2$, we choose a plane Π such that $\Pi(X_i^\beta) \neq \Pi(Y^\beta)$, $\beta \in Q$, where, by $\Pi(z)$, we mean the projection of the point $z \in R^\nu$ onto the phase Π. Such a plane is found the virtue of the finiteness of the number of pursuers n. If the problem of the weak evasion of the projections of the evaders from the projections of the pursuers is solvable, then the initial problem is also thereby solvable. Next, in section, we consider $\nu = 2$.

We shall denote the c-coordinate of the vector $z \in R^\nu$ by z_c.

We define the functions

$$l_c(t)\{e_c(t)\} - \text{amount } \alpha \in I : x_{c\alpha}^{(q-1)}(t) < \{=\} y_c^{(q-1)}(t)$$

for all $t \geq 0$ and introduce the positive constants δ_c, ρ_c, γ_c such that

$$\sqrt{\gamma_1^2 + \gamma_2^2} \leq \gamma, \quad \gamma_c = \delta_c + 2\rho_c n + \rho_c/4, \quad \delta_c - \rho_c/4 > 0 \tag{2.1}$$

We define a set

$$\Omega_c(t) = \{\delta_c + 2\rho_c l_c(t) + 2\rho_c k, k = 0, 1, \ldots, e_c(t)\}$$

and the quantity $\omega_c(t) \in \Omega_c(t)$ for each instant $t \in [0, \infty)$ as follows: if $e_c(t) = 0$, then $\omega_c(t) = \delta_c + 2\rho_c l_c(t)$; if $e_c(t) \geq 1$, then $\omega_c(t)$ is determined from the condition

$$\min_{\alpha \in E_c(t)} \{|\omega_c(t) - x_{c\alpha}^{(q)}(t)|\} = \max_{\omega \in \Omega_c(t)} \min_{\alpha \in E_c(t)} \{|\omega - x_{c\alpha}^{(q)}(t)|\} \geq \rho_c \tag{2.2}$$

where $E_c(t) = \{\beta \in I: x_{c\beta}^{(q-1)}(t) = y_c^{(q-1)}(t)\}$ and, consequently, $|E_c(t)| = e_c(t)$.

Estimate (2.2) follows from a well-known result [5, Lemma 2.1].

To be specific: if several values of $\omega_c(t)$ exist, we take the largest of them. The quantity $\omega_c(t)$ is therefore uniquely defined for all $t \geq 0$ and

$$\omega_c(t) \in \Omega_c^* = \{\delta_c + 2\rho_c k, k = 0, 1, \ldots, n\} \tag{2.3}$$

We denote a sphere of radius ρ with its centre at the point o by $\mathcal{D}(o, \rho)$ and also introduce the notation

$$a^{[k]} = \frac{a^k}{k!}, \quad \sum_{ir}^{x}(t, T) = \sum_{k=0}^{p-r-1} x_i^{(r+k)}(t) T^{[k]}, \quad \sum_{r}^{y}(t, T) = \sum_{k=0}^{q-r-1} y^{(r+k)}(t) T^{[k]}$$

Lemma 1. The following holds for all $t \geq 0$, $T > 0$ and $r \in Q$:

(1) the attainability domain $x_i^{(r)}$ at the instant $t + T$ coincides with the set

$$\mathcal{D}\left(\sum_{ir}^{x}(t, T), T^{[p-r]}\right)$$

(2) Suppose $\upsilon_c(\tau) = \upsilon_c(t)$ for all $\tau \in [t, t + T]$. Then,

$$y_c^{(r)}(t + T) = \sum_{cr}^{y}(t, T) + \upsilon_c(t) T^{[q-r]}$$

We define the functions $T_{cir}(t) \geq 0$ as the time after which the c-coordinates $x_i^{(r)}$ and $y^{(r)}$ can be identical for the first time, that is, the equality

$$\Delta_{ci}^{(r)}(t + T_{cir}(t)) = 0, \text{ where } \Delta_i(t) = y(t) - x_i(t)$$

can be satisfied subject to the condition that $\upsilon_c(\tau) = \upsilon_c(t)$ for all $\tau \in [t, \infty)$. It follows from Lemma 1 that the value of $T_{cir}(t)$ for all $t \in [0, \infty)$ and $r \in Q$ is determined as the least non-negative (with respect to T) root of the polynomial

$$-\operatorname{sign}(\Delta_{ci}^{(r)}(t))T^{[p-r]} - \sum_{k=q-r+1}^{p-r-1} x_{ci}^{(r+k)}(t)T^{[k]} + \\ + (\upsilon_c(t) - x_{ci}^{(q)}(t))T^{[q-r]} + \sum_{k=1}^{q-r-1} \Delta_{ci}^{(r+k)}(t)T^{[k]} + \Delta_{ci}^{(r)}(t) = 0 \tag{2.4}$$

Such a root exists since Eq. (2.4) can be represented in the form

$$T^{p-r} + a_1 T^{p-r-1} + \ldots + a_{p-r-1}T = a_{p-r}, \quad \text{where} \quad a_{p-r} \geq 0$$

Suppose

$$T_{cr}(t) = \min\{T_{c1r}(t), T_{c2r}(t), \ldots, T_{cnr}(t)\} \tag{2.5}$$

We define the functions

$$\xi_{cir}^{\pm}(t) = \sum_{cr}^{y}(t, T_{cir}(t)) + \left(\upsilon_c(t) \pm \frac{\rho_c}{8}\right)T_{cir}^{[q-r]}(t) - \sum_{cir}^{x}(t, T_{cir}(t)) \mp T_{cir}^{[p-r]}(t)$$

$$K_{cir}(t) = \begin{cases} 1, \text{ if } \Delta_{ci}^{(r)}(t) < 0 \text{ and } \xi_{cir}^{+}(t) \geq 0 \\ -1, \text{ if } \Delta_{ci}^{(r)}(t) > 0 \text{ and } \xi_{cir}^{-}(t) \leq 0 \\ 0 \text{ otherwise} \end{cases} \tag{2.6}$$

for all $t \in [0, \infty)$ and $r \in Q$.

Lemma 2. Suppose the evader E_1 uses an arbitrary constant control. Then, the following assertions hold for any permissible control $u_i(t)$ of a pursuer P_i and $r \in Q$.

1. If, for $t > 0$ and a certain $\sigma > 0$ when $\tau \in [t - \sigma, t)$,

$$\Delta_{ci}^{(r)}(\tau) < 0\{>0\}, \quad \Delta_{ci}^{(r)}(t) = 0, \quad \Delta_{di}^{(r)}(\tau) \neq 0, \quad \Delta_{di}^{(r)}(t) \neq 0 \tag{2.7}$$

then a $\varepsilon \in (0, \sigma]$ is found such that, for $\tau \in [t - \varepsilon, t)$,

$$K_{cir}(\tau) = 1\{=-1\}, \quad T_{dir}(\tau) > T_{cir}(\tau), \quad d \in \{1, 2\} \backslash \{c\}$$

2. If, for $t > 0$ and a certain $\sigma > 0$ when $\tau \in [t - \sigma, t)$,

$$\Delta_{ci}^{(r)}(\tau) \neq 0, \quad \Delta_i^{(r)}(t) = 0$$

then $\varepsilon \in (0, \sigma]$ is found such that, for $\tau \in [t - \varepsilon, t)$,

$$K_{1ir}(\tau) \neq 0, \quad T_{2ir}(\tau) \geq T_{1ir}(\tau) > 0 \quad \text{or} \quad K_{2ir}(\tau) \neq 0, \quad T_{1ir}(\tau) \geq T_{2ir}(\tau) > 0$$

Proof. The continuity of the functions $T_{1ir}(\tau)$, $T_{2ir}(\tau)$, for all $\tau \in [0, \infty)$ follows from relation (2.4) and the conditions of the lemma.

1. Suppose that, when the "less than" sign is chosen in the first of them, relations (2.7) are satisfied. In this case, $T_{cir}(t) = 0$, $T_{dir}(t) > 0$ and, on the taking account of the continuity of these functions, we obtain that a $\varepsilon \in (0, \sigma]$ exists such that

$$T_{dir}(\tau) > T_{cir}(\tau), \quad \frac{\rho_c}{8} \geq 2\frac{(q-r)!}{(p-r)!}T_{cir}^{p-q}(\tau), \quad \tau \in [t-\varepsilon, t)$$

It follows from the definition of (2.6) that $K_{cir}(\tau) = 1$, $\tau \in [t, -\varepsilon, t)$ if $\xi_{cir}^{+}(\tau) \geq 0$ which is equivalent to the inequality

$$\left(\left(\sum_{cr}^{y}(\tau, T_{cir}(\tau)) + \upsilon_c(\tau)T_{cir}^{[q-r]}(\tau)\right) - \left(\sum_{cir}^{x}(\tau, T_{cir}(\tau)) - T_{cir}^{[p-r]}(\tau)\right)\right) +$$
$$+\left(\frac{\rho_c}{8} - 2\frac{(q-r)!}{(p-r)!}T_{cir}^{p-q}(\tau)\right)T_{cir}^{[q-r]}(\tau) \geq 0$$

which is satisfied since the first term is equal to zero by virtue of the definition of the function $T_{cir}(\tau)$ and the second term is non-negative according to the choice of ε. The remaining case is considered in a similar way.

2. Taking account of the continuity of these functions $T_{cir}(\tau) > 0$, $\tau \in [t-\sigma, t)$, $T_{cir}(t) = 0$, a $\varepsilon \in (0, \sigma]$ exists such that

$$\left(\frac{\rho_c(p-r)!}{16(q-r)!}\right)^{1/(p-q)} \geq T_{\alpha ir}(\tau) \geq T_{\beta ir}(\tau) > 0; \quad \alpha = 2, \quad \beta = 1 \quad \text{or} \quad \alpha = 1, \quad \beta = 2$$

for all $\tau \in [t-\varepsilon, t)$.

In a similar manner to Assertion 1, it is proved for such ε that $K_{cir}(\tau) \neq 0$ when $\tau \in [t-\varepsilon, t)$.

We will now define the functions

$$J_{cir}(t) = \min\{T_{d\alpha\beta}(t) : (d, \alpha, \beta) \in \{1, 2\} \times I \times Q \text{ and } (d, \alpha, \beta) \neq (c, i, r)\}$$

$$B_{cir}(t) = \begin{cases} 1, \text{ if } K_{cir}(t) \neq 0, \quad J_{cir}(t) \geq T_{cir}(t) = T_{cr}(t) \\ B_{1\alpha\beta}(t) = B_{2\alpha\beta}(t) = 0, \text{when } c = 2 \text{ and } B_{1\alpha r}(t) = 0 \\ \text{for all } \alpha \in I, \quad \beta \in \{r+1, r+2, \ldots, q-1\} \\ 0 \text{ in the remaining cases} \end{cases}$$

for all $t \geq 0$ and $r \in Q$.

It follows from Lemma 2 that no more than one of the $2q$-functions B_{cir} becomes equal to unity in the case of fixed i at each instant $t \geq 0$.

We define the functions $\upsilon_c(t)$ as follows:

$$\upsilon_c(t) = \begin{cases} \omega_c(\tau_{2b}^c), & t \in [\tau_{2b}^c, \tau_{2b+1}^c) \\ \omega_c(\tau_{2b+1}^c) + K_{c\alpha r}(\tau_{2b+1}^c)\rho_c/4, & t \in [\tau_{2b+1}^c, \tau_{2b+2}^c) \end{cases} \tag{2.8}$$

where $\tau_{2b+1}^c \geq \tau_{2b}^c$ is the instant when $\alpha \in I$, $r \in Q$ are first found such that

$$B_{c\alpha r}(\tau_{2b+1}^c) = 1, \quad \upsilon_d(\tau_{2b+1}^c) \in \Omega_d^* \tag{2.9}$$

and $\tau_{2b+2}^c > \tau_{2b+1}^c$ is the instant when, for the first time, for at least one $\beta \in I$,

$$\Delta_{c\beta}^{(r)}(\tau_{2b+2}^c) = 0 \tag{2.10}$$

Here, $\tau_0^c = 0$, $d \in \{1, 2\}\backslash\{c\}$, $b = 0, 1, 2, \ldots$ and, to be specific: if several $\alpha \in I$ are found which satisfy the property (2.9), then we take the least of them.

We now define the sequence $t_b^c : t_0^c = 0$; $\tau_{2k+1}^c > t_{b-1}^c$ is the instant when $r = q-1$ in algorithm (2.8)–(2.10) for the first time and then $t_b^c = \tau_{2k+2}^c$ $(b = 1, 2, \ldots)$.

Henceforth, we shall assume that the control $\upsilon(t) = (\upsilon_1(t), (\upsilon_2(t))^T$ and the sequence $\{\tau_b^c\}_{b=0}^{b_c}$ are defined according to algorithm (2.8)–(2.10) where either $b_c < \infty$ or $b_c = \infty$ and the sequence $\{\tau_b^c\}_{b=0}^{b_c^*} \subset \{\tau_b^c\}_{b=0}^{b_c}$ is define as described above where either $b_c^* < \infty$ or $b_c^* = \infty$.

Lemma 3. The following assertions hold for any permissible controls $u_i(t)$.

1. If $b_c \geq 2$, then $\{\tau_{2b}^1\}_{b=1}^{b_1^2} \cap \{\tau_{2b}^2\}_{b=1}^{b_2^2} = \varnothing$ and $\Delta_{ci}^{(r)}(t) \neq \varnothing$ for all $r \in Q$, $t \in (\tau_{2b}^c, \tau_{2b+2}^c)$, $b = 0, 1, \ldots, b_c^2 - 1$ where $b_c^2 = \text{enter}[b_c/2]$.
2. The inclusion $\upsilon_c(\tau) \in [\delta_c - \rho_c/4, \gamma_c]$, where $\tau \in \{\tau_b^c\}_{b=0}^{b_c}$, holds.
3. If $b_c = \infty$, then, also $b_c^* = \infty$,
4. $\upsilon_c(t_b^c) - \rho_c/4 \leq \upsilon_c(t) \leq \upsilon_c(t_b^c) + \rho_c/4$ for all $t \in [t_b^c, t_{b+1}^c)$.

Proof. Assertion 1 follows from a well-known result [5, Lemma 3.2].
Assertion 2 follows from the fact that, according to relations (2.3) and (2.8)–(2.10),

$$\upsilon_c(\tau_{2b}^c) \in \Omega_c^* \subset [\delta_c, \delta_c + 2\rho_c n], \quad \upsilon_c(\tau_{2b+1}^c) = \upsilon_c(\tau_{2b}^c) \pm \rho_c/4 \in [\delta_c - \rho_c/4, \gamma_c]$$

We will now prove Assertion 3. If $q = 1$, then $\{t_b^c\}_{b=0}^{b_c^*} = \{\tau_{2b}^c\}_{b=0}^{\infty}$, whence $b_c^* = \infty$. Suppose $q = 2$. Let us assume that, contrary to the assertion, $b_c = \infty$, $b_c^* < \infty$, and a number N is then found such that, for any $b \geq N$, we have $B_{c\alpha 1}(\tau_{2b+1}^c) = 0$, $\alpha \in I$ and that, for just one $\beta \in I$, $B_{c\beta 0}(\tau_{2b+1}^c) = 0$. It follows from Assertion 1, without loss in generality, that a number $k \in \{0, 1, \ldots, n\}$ is found such that

$$\dot{x}_{c1}(t), \dot{x}_{c2}(t), \ldots, \dot{x}_{ck}(t) < \dot{y}_c(t) < \dot{x}_{ck+1}(t), \dot{x}_{ck+2}(t), \ldots, \dot{x}_{cn}(t)$$

for all $t \geq \tau_{2(N+1)}^c$.
If follows from the last fact that a number M exists such that

$$x_{c1}(t), x_{c2}(t), \ldots, x_{ck}(t) < y_c(t) < x_{ck+1}(t), x_{ck+2}(t), \ldots, x_{cn}(t)$$

for all $t \geq \tau_{2(N+M)}^c$.
On combining the two inequalities for $t \geq \tau_{2(N+M)}^c$, we obtain $b_c < \infty$. This contradiction completes the proof. The case when $q \geq 3$ is treated in a similar manner.
We will now prove Assertion 4. Suppose

$$t_b^c = \tau_{2N}^c \leq \tau_{2N+1}^c < \tau_{2(N+1)}^c \leq \tau_{2(N+1)+1}^c < \ldots < \tau_{2(N+M)}^c = t_{b+1}^c$$

Then, using Assertion 1 and algorithm (2.8)–(2.10), we obtain

$$\upsilon_c(t_b^c) = \upsilon_c(\tau_{2N}^c), \quad \upsilon_c(\tau_{2N+1}^c) = \upsilon_c(t_b^c) \pm \rho_c/4, \quad \upsilon_c(\tau_{2(N+1)}^c) = \upsilon_c(t_b^c), \ldots$$
$$\upsilon_c(\tau_{2(N+M-1)}^c) = \upsilon_c(t_b^c), \quad \upsilon_c(\tau_{2(N+M-1)+1}^c) = \upsilon_c(t_b^c) \pm \rho_c/4$$

whence it follows that Assertion 4 holds.

We will now prove that algorithm (2.8)–(2.10) determines $\upsilon(t) = (\upsilon_1(t), \upsilon_2(t))^T$ for all $t \in [0, \infty)$. To do this, it is sufficient to prove the following lemma.

Lemma 4. For any set of permissible controls $u_i(t)$ of the pursuers P_i either the value of b_c is finite or $\lim \tau_b^c = \infty$ when $b \to \infty$.

Proof. Consider the case when $c = 1$. Only one of two cases is possible for each set of permissible controls $u_i(t)$.

Case 1. Algorithm (2.8)–(2.10) is applied a finite number of times and therefore the value of b_1 is finite.

Case 2. Algorithm (2.8)–(2.10) is applied an infinite number of times. It is required to show that the sequence $\{\tau_b^1\}_{b=0}^{\infty}$ obtained using this formula possesses the following property: $\lim \tau_b^c = \infty$ when $b \to \infty$. Let us assume that the opposite is true: a set of permissible controls $u_i^*(t)$ exists such that

$$\lim \tau_b^1 = \tau^* < \infty \text{ when } b \to \infty$$

1. We consider the numbers $x_{1i}^{(q-1)}(\tau^*)$. Suppose they take $r \in l$ different values $\xi_1 < \xi_2 < \ldots < \xi_0$. Without loss of generality, we will assume that

$$x_{1s}^{(q-1)}(\tau^*) = \xi_k, \quad s \in S_k, \quad \text{where}$$
$$S_k = \{s_{k-1}+1, s_{k-1}+2, \ldots, s_k\}, \quad k = 1, 2, \ldots, r \quad (s_0 = 0, s_r = n)$$

For each $\varepsilon = [0, \tau^*]$, we define the sets

$$H_k(\varepsilon) = \bigcup_{s \in S_k} \{z \in R^1 : z = x_{1s}^{(q-1)}(t), t \in [\tau^* - \varepsilon, \tau^*]\}, \quad k = 1, 2, \ldots, r$$

Suppose $G_1, G_2 \subset R^1$. We will use the notation

$$\operatorname{dist}(G_1, G_2) = \inf_{g_1 \in G_1, g_2 \in G_2} |g_1 - g_2|$$
$$h(\varepsilon) = \min\{\operatorname{dist}(H_k(\varepsilon), H_{k+1}(\varepsilon)), k = 1, 2, \ldots, r-1\}$$
$$H(\varepsilon) = h(\varepsilon) - 2\gamma_1\varepsilon, \quad \varepsilon \in [0, \tau^*]$$

By virtue of the continuity of the function $H(\varepsilon)$ and the condition $h(0) > 0$, we obtain that a $\varepsilon_1 > 0$ exists such that $H(\varepsilon) > 0$ for all $\varepsilon \in [0, \varepsilon_1]$ and, from this, that

$$h(\varepsilon)/\gamma_1 > 2\varepsilon \quad \text{for all} \quad \varepsilon \in [0, \varepsilon_1] \tag{2.11}$$

2. If $|S_k| = 1$, we put $\varepsilon_2^k = \infty$. Suppose $|S_k| \geq 2$ and $\alpha, \beta \in S_k$.
We note that

$$x_{1\alpha}^{(q-1)}(\tau^*) = x_{1\beta}^{(q-1)}(\tau^*) = \xi_k \tag{2.12}$$

Using the notation $T = [\tau^* - \varepsilon, \tau^*)$, $\bar{T} = [\tau^* - \varepsilon, \tau^*]$, we pick out all possible cases of the mutual ordering of the values $x_{1\alpha}^{(q-1)}, x_{1\beta}^{(q-1)}, x_{1\alpha}^{(q)}, x_{1\beta}^{(q)}$.

(2.1) $x_{1\alpha}^{(q)}(\tau^*) > x_{1\beta}^{(q)}(\tau^*)$ and, by virtue of the continuity of these functions, and $\varepsilon > 0$ exists such that

$$x_{1\alpha}^{(q)}(t) > x_{1\beta}^{(q)}(t), \quad t \in \bar{T}$$

Furthermore, when account is taken of equality (2.12), we have

$$x_{1\alpha}^{(q-1)}(t) < x_{1\beta}^{(q-1)}(t), \quad t \in T$$

(2.2) $x_{1\alpha}^{(q)}(\tau^*) < x_{1\beta}^{(q)}(\tau^*)$ and, in the same way as in case (2.1), an $\varepsilon > 0$ exists, such that

$$x_{1\alpha}^{(q)}(t) < x_{1\beta}^{(q)}(t), \quad t \in \bar{T}, \quad x_{1\alpha}^{(q-1)}(t) > x_{1\beta}^{(q-1)}(t), \quad t \in T$$

(2.3) $x_{1\alpha}^{(q)}(\tau^*) = x_{1\beta}^{(q)}(\tau^*)$ and this case has several versions:

(2.3.1) an $\varepsilon > 0$ exists such that $x_{1\alpha}^{(q)}(t) = x_{1\beta}^{(q)}(t)$, $t \in \bar{T}$ and, then, $x_{1\alpha}^{(q-1)}(t) = x_{1\beta}^{(q-1)}(t)$, $t \in \bar{T}$ also.

(2.3.2) an $\varepsilon > 0$ exists such that $x_{1\alpha}^{(q)}(t) > x_{1\beta}^{(q)}(t)$, $t \in T$ and, then, as in case 2.1, $x_{1\alpha}^{(q-1)}(t) < x_{1\beta}^{(q-1)}(t)$, $t \in T$.

(2.3.3) an $\varepsilon > 0$ exists such that $x_{1\alpha}^{(q)}(t) < x_{1\beta}^{(q)}(t)$, $t \in T$ and, then, as in case 2.2, $x_{1\alpha}^{(q-1)}(t) > x_{1\beta}^{(q-1)}(t)$, $t \in T$.

Now, on picking out all the $x_{1s}^{(q-1)}, x_{1s}^{(q)}$, $s \in S_k$ in pairs as $x_{1\alpha}^{(q-1)}, x_{1\beta}^{(q-1)}, x_{1\alpha}^{(q)}, x_{1\beta}^{(q)}$, we obtain that a $\varepsilon_2^k > 0$ exists such that the natural ordering of $x_{1s}^{(q-1)}$ and $x_{1s}^{(q)}$, $s \in S_k$ does not change in the interval $[\tau^* - \varepsilon_2^k, \tau^*)$. Without loss of generality, this last fact means that

$$\left.\begin{matrix} x_{1s_{k-1}+1}^{(q-1)}(t)\{<=\}x_{1s_{k-1}+2}^{(q-1)}(t)\ldots\{<=\}x_{1s_k}^{(q-1)}(t) \\ x_{1s_{k-1}+1}^{(q)}(t)\{>=\}x_{1s_{k-1}+2}^{(q)}(t)\ldots\{>=\}x_{1s_k}^{(q)}(t) \end{matrix}\right\}, \quad t \in [\tau^* - \varepsilon_2^k, \tau^*) \tag{2.13}$$

Here, $\{<=\}$, $\{>=\}$ means that, either the $<(>)$ sign or the $=$ sign is chosen in the first (second) row of the formula over the whole interval $[\tau^* - \varepsilon_2^k, \tau^*)$.

We choose $\varepsilon_2 = \min\{\varepsilon_2^1, \varepsilon_2^2, \ldots, \varepsilon_2^r,\} > 0$.

3. It follows from the continuity of $x_{1i}^{(q)}(t)$ that a $\varepsilon_3^i > 0$ exists such that

$$\left|x_{1i}^{(q)}(\tau^* - \varepsilon') - x_{1i}^{(q)}(\tau^* - \varepsilon'')\right| < \rho_1/4 \quad \text{for all} \quad \varepsilon', \varepsilon'' \in [0, \varepsilon_3^i) \tag{2.14}$$

We now take $\varepsilon_3 = \min\{\varepsilon_3^1, \varepsilon_3^2, \dots, \varepsilon_3^n\} > 0$.

4. We define

$$\varepsilon^* = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\} > 0 \tag{2.15}$$

It follows from the assumption of the existence of a finite limit of the sequence $\{\tau_b^1\}_{b=0}^{\infty}$ that, up to the instant $\tau^* - \varepsilon^* < \tau^*$, the control $\upsilon_1(t)$ is defined and a number N exists such that $t_N^1, t_{N+1}^1, \dots \in [\tau^* - \varepsilon^*, \tau^*]$ where, according to Lemma 3, $\{t_b^1\}_{b=0}^{\infty} \subset \{\tau_b^1\}_{b=0}^{\infty}$.

We will now consider a game Γ starting from the instant $\tau^* - \varepsilon^*$ and we will prove that a number M is found: $t_{(N+M)}^1 > \tau^*$ and by this means we obtain a contradiction to the supposition concerning a finite value of $\lim \tau_b^1$ when $b \to \infty$, and the lemma will be proved.

So, the instant $t_N^1 \in [\tau^* - \varepsilon^*, \tau^*)$. It is necessary that $y_1^{(q-1)}(t_N^1) \in H_k(\varepsilon^*)$ for a certain $k \in \{1, 2, \dots, r\}$. We recall that

$$x_{1s}^{(q-1)}(t) \in H_k(\varepsilon^*), \quad t \in [\tau^* - \varepsilon^*, \tau^*], \quad s \in S_k$$

and that just one $\alpha \in S_k$ exist such that $y_1^{(q-1)}(t_N^1) = x_{1\alpha}^{(q-1)}(t_N^1)$.

If follows from condition (2.2) that two cases are possible.

(4.1) $\upsilon_1(t_N^1) \geq x_{1\alpha}^{(q)}(t_N^1) + \rho_1$, where α is one of the several successive subscripts from S_k.

It follows from the Lemma 3 that

$$\upsilon_1(t_N^1) - \rho_1/4 \leq \upsilon_1(t) \leq \upsilon_1(t_N^1) + \rho_1/4, \quad t \in [t_N^1, t_{N+1}^1)$$

Taking account of relations (2.14) and (2.15), we conclude from this that

$$\upsilon_1(t) > x_{1\alpha}^{(q)}(t) + \rho_1/2 \quad \text{for all} \quad t \in [t_N^1, t_{N+1}^1) \tag{2.16}$$

According to relation (2.13), one of the following two equalities must be satisfied at the instant t_{N+1}^1:

(4.1.1) $y_1^{(q-1)}(t_{N+1}^1) = x_{1\alpha}^{(q-1)}(t_{N+1}^1)$ and this case is impossible by virtue of inequality (2.16);

(4.1.2) $y_1^{(q-1)}(t_{N+1}^1) = x_{1\beta}^{(q-1)}(t_{N+1}^1)$, $\beta > \alpha$ (β is one or several successive subscripts from S_k).

So, the case 4.1.2 remains. Consider the system of inequalities

$$\left|x_{1\beta}^{(q)}(t_{N+1}^1) - x_{1\beta}^{(q)}(t)\right| < \frac{\rho_1}{4}, \quad \upsilon_1(t) - \frac{\rho_1}{2} > x_{1\alpha}^{(q)}(t) > x_{1\beta}^{(q)}(t), \quad t \in [t_n, t_{N+1}) \tag{2.17}$$

The correctness of the first inequality follows from relations (2.14) and (2.15), and the correctness of the second chain of inequalities follows from relations (2.16) an d(2.13). From system (2.17), we obtain that

$$\upsilon_1(t) > x_{1\beta}^{(q)}(t_{N+1}^1) + \rho_1/4, \quad t \in [t_N, t_{N+1})$$

and, therefore, the value of $\upsilon_1(t_{N+1}^1)$, according to algorithm (2.8)–(2.10), will be defined so that

$$\upsilon_1(t_{N+1}^1) \geq x_{1\beta}^{(q)}(t_{N+1}^1) + \rho_1$$

Continuing further, we obtain that an instant t_{N+L}^1 exists such that

$$y_1^{(q-1)}(t_{N+L}^1) = x_{1s_k}^{(q-1)}(t_{N+L}^1), \quad \upsilon_1(t_{N+L}^1) \geq x_{1s_k}^{(q)}(t_{N+L}^1) + \rho_1$$

from which we obtain that

$$x_{1s}^{(q-1)}(t) < y_1^{(q-1)}(t), \quad t \in (t_{N+L}^1, \tau^*], \quad s \in S_k$$

This means that, in order for $t_{N+L+1}^1 \in [\tau^* - \varepsilon^*, \tau^*)$, it is necessary that the following inequality be satisfied

$$y_1^{(q-1)}(t_{N+L+1}^1) = x_{1\eta}^{(q-1)}(t_{N+L+1}^1), \quad \eta \in I \backslash S_k$$

and this means that the value of $y_1^{(q-1)}$ from the set $H_k(\varepsilon^*)$ must fall within the set $H_{k+1}(\varepsilon^*)$. It follows from equality (2.11) that, even in the case of the maximum value of υ_1 which, according to Lemma 3, is equal to γ_1, a time greater than $2\varepsilon^*$ is required for this, from which $t_{N+L+1}^1 - t_{N+L}^1 > 2\varepsilon^*$. So, a number $M = L + 1$: $t_{N+M}^1 > \tau^*$ exists.

(4.2) $\upsilon_1(t_N^1) \le x_{1\alpha}^{(q)}(t_N^1) - \rho_1$. The existence of the number M is proved in the same way. The case $c = 2$ is treated similarly. The lemma is proved.

It follows from Lemma 3 and 4 that the inclusion

$$\upsilon_c(t) \in [\delta_c - \rho_c/4, \gamma_c] \quad \text{for all} \quad t \in [0, \infty) \tag{2.18}$$

is satisfied in the case of the functions υ_c.

The strategy of the evader E_1 is therefore determined; at each instant of time $t \ge 0$, the evader E_1 determines the functions $\upsilon_1(t)$ and $\upsilon_2(t)$ and thereby completely defines his control $\upsilon(t)$.

Theorem 1. In game Γ, when $m = 1$, weak evasion from any initial positions occurs.

Proof. We will now prove that the strategy of the evader, defined by algorithm (2.8)–(2.10), is a eak evasion strategy. Actually, the control υ belongs to the class of piecewise-constant functions and changes in value at the instants $\{\tau_b^1\}_{b=0}^{b_1} \cup \{\tau_b^2\}_{b=0}^{b_2}$. Using relations (2.18) and (2.1), we obtain

$$\|\upsilon(t)\| \le \sqrt{\gamma_1^2 + \gamma_2^2} \le \gamma$$

The satisfaction of the condition $x_i^{(r)}(t) \ne y^{(r)}(t)$ for all $r \in Q$ and $t \ge 0$ follows from Lemmas 3 and 4.

3. THE CASE WHEN $m \ge 2$

We will now define a weak evasion strategy for a group of coordinated evaders E_j.

Theorem 2. In game Γ, a weak evasion occurs from any initial positions.

Proof. We define a game Γ_1 of $nm + 1$ players in the space R^ν: nm pursuers P_i^j and an evader with laws of motion and initial conditions (when $t = 0$)

$$\begin{aligned} &x_{ij}^{(p)} = u_i, \quad \|u_i\| \le 1; \quad y^{(q)} = w, \quad \|w\| \le \gamma \\ &x_{ij}^{(\beta)}(0) = X_i^\beta - Y_j^\beta, \quad x_{ij}^{(\alpha)}(0) = X_i^\alpha, \quad y^{(\beta)}(0) = 0, \quad \beta \in Q, \quad \alpha \in P\backslash Q \end{aligned} \tag{3.1}$$

For all permissible controls u_i, w, numbers $r \in Q$ and $t \ge 0$, we have

$$x_{ij}^{(r)}(t) = \sum_{k=r}^{p-1} x_{ij}^{(k)}(0) t^{[k-r]} + \int_0^t (t-\tau)^{[p-r-1]} u_i(\tau) d\tau =$$

$$= \sum_{k=r}^{q-1} (X_i^k - Y_j^k) t^{[k-r]} = \sum_{k=q}^{p-1} X_i^k t^{[k-r]} + \int_0^t (t-\tau)^{[p-r-1]} u_i(\tau) d\tau$$

$$y^{(r)}(t) = \int_0^t (t-\tau)^{[q-r-1]} w(\tau) d\tau$$

In game Γ_1, the pursuers act in the following way: at each instant of time $t \in [0, \infty)$, each of the pursuers P_i^j uses the same control $u_i(t)$, chosen by the pursuer P_i in the game Γ. In this case, the following equality holds

$$x_i^{(r)}(t) = x_{ij}^{(r)}(t) + \sum_{k=r}^{q-1} Y_j^k t^{[k-r]} \tag{3.2}$$

Suppose $w(t)$ is a control which ensures a weak evasion in the game Γ_1, which has been chosen by the evader at the instant of time t. Then,

$$x_{ij}^{(r)}(t) \ne y^{(r)}(t) \tag{3.3}$$

The existence of such a control follows from Theorem 1.

We will now determine the control of the evaders E_j at each instant of time $t \geq 0$ in the following way: $\upsilon(t) = w(t)$. In this case,

$$y_j^{(r)}(t) = y^{(r)}(t) + \sum_{k=r}^{q-1} Y_j^k t^{[k-r]} \tag{3.4}$$

Combining relations (3.2), (3.3) and (3.4), we obtain that

$$x_i^{(r)}(t) \neq y_j^{(r)}(t), \quad t \in [0, \infty)$$

This research was supported by the Federal Agency for Education (A04-2.8-60) and the "Universities of Russia" programme (34126).

REFERENCES

1. SATIMOV, N. Yu. and RIKHSIYEV, B. B., Quasilinear differential evasion games. *Differents. Uravneniya*, 1978, **14**, 6, 1046–1052.
2. SATIMOV, N. Yu. and RIKHSIYEV, B. B., *Methods of Solving the Evasion Problem in Mathematical Control Theory*. Fan, Tashkent, 2000.
3. CHIKRII, A. A., *Conflict-Control Processes*. Naukova Dumka, Kiev, 1992.
4. GRIGORENKO, N. L., *Mathematical Methods of Controlling Certain Dynamic Processes*. Izd. MGU, 1990.
5. BLAGODATSKIKH, A. I., Evasion of strictly coordinated evaders in a problem of group pursuit. *Izv. Inst. Matematiki i Informatiki, Udmurt. Gos. Univ.*, 2004, 2, 3–4.

Translated by E.L.S.